

A variant of forward-backward splitting method for the systems of inclusion problems

R. Díaz Millán*

January 5, 2016

Abstract

In this paper, we propose variants of forward-backward splitting method for solving the system of splitting inclusion problem. We propose a conceptual algorithm containing three variants, each having a different projection steps. The algorithm consists in two parts, the first and main contains an explicit Armijo-type search in the spirit of the extragradient-like methods for variational inequalities. In the iterative process the operator forward-backward is computed only one time for each inclusion problem, this represent a great computational saving if we compare with Tseng's algorithm, because the computational cost of this operator is very high. The second part of the scheme consists in special projection steps. The convergence analysis of the proposed scheme is given assuming monotonicity on both operators, without assuming Lipschitz continuity on the forward operators.

Keywords: Armijo-type search, Maximal monotone operators, Splitting methods, Systems of inclusion problems

Mathematical Subject Classification (2008): 90C47, 49J35.

1 Introduction

First, we introduce a notation and some definitions. The inner product in \mathbb{R}^n is denoted by $\langle \cdot, \cdot \rangle$ and the norm induced by the inner product by $\| \cdot \|$. We denote by 2^C the power set of C . For X a nonempty, convex and closed subset of \mathbb{R}^n , we define the orthogonal projection of x onto X by $P_X(x)$, as the unique point in X , such that $\|P_X(x) - y\| \leq \|x - y\|$ for all $y \in X$. Let $N_X(x)$ be the normal cone to X at $x \in X$, i.e., $N_X(x) = \{d \in \mathbb{R}^n : \langle d, x - y \rangle \geq 0 \ \forall y \in X\}$. Recall that an operator $T : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ is monotone if, for all $(x, u), (y, v) \in Gr(T) := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^n : u \in T(x)\}$, we have $\langle x - y, u - v \rangle \geq 0$, and it is maximal if T has no proper monotone extension in the graph inclusion sense.

In this paper, we propose a modified algorithm for solving a system of splitting inclusion problem, for the sum of two operators. Given a finite family of pair of operators $\{A_i, B_i\}_{i \in \mathbb{I}}$, with $\mathbb{I} =: (1, 2, \dots, m)$ and $m \in \mathbb{N}$. The system of inclusion problem consists in:

$$\text{Find } x \in \mathbb{R}^n \text{ such that } 0 \in (A_i + B_i)(x) \text{ for all } i \in \mathbb{I}, \quad (1)$$

*Federal Institute of Education, Science and Technology, Goiânia, Brazil, e-mail: rdiazmillan@gmail.com

where the operators $A_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are point-to-point and monotone and the operators $B_i : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ are point-to-set maximal monotone operators. The solution of the problem is given by the interception of the solution of each component of the system, i.e., $S_* = \cap_{i \in \mathbb{I}} S_*^i$, where S_*^i is defined as $S_*^i := \{x \in \mathbb{R}^n : 0 \in A_i(x) + B_i(x)\}$.

The problem (1) is a generalization of the system of variational inequalities, introduced by I.V. Konnov in [16], taking the operators $B_i = N_{C_i}$ for all $i \in \mathbb{I}$, which have been studied in [9–11, 17]. A generalization of this results have been studied in [14, 19], where the hypothesis that all A_i are Lipschitz continuous for all $i \in \mathbb{I}$, is assumed for the convergence analysis. In this paper we improve this result assuming only monotonicity for all operators A_i , and maximal monotonicity for the operators B_i . Also, we improve the linesearch proposed by Tseng in [21], calculating in each tentative of find the step size, the operator forward-backward only one time for each inclusion problem of the system. This improves the algorithm in the computational sense, because this operator is very expensive to compute. The idea for this manuscript was motivated from the works [6, 12].

Problem (1) have many applications in operations research, mathematical physics, optimization and differential equations. This kind of problem have been deeply studied and has recently received a lot attention, due to the fact that many nonlinear problems, arising within applied areas, are mathematically modeled as nonlinear operator system of equations and/or inclusions, which each one are decomposed as sum of two operators.

2 Preliminaries

In this section, we present some definitions and results needed for the convergence analysis of the proposed algorithm. First, we state two well-known facts on orthogonal projections.

Proposition 2.1 *Let X be any nonempty, closed and convex set in \mathbb{R}^n , and P_X the orthogonal projection onto X . For all $x, y \in \mathbb{R}^n$ and all $z \in X$ the following hold:*

- (i) $\|P_X(x) - P_X(y)\|^2 \leq \|x - y\|^2 - \|(P_X(x) - x) - (P_X(y) - y)\|^2$.
- (ii) $\langle x - P_X(x), z - P_X(x) \rangle \leq 0$.
- (iii) $P_X = (I + N_X)^{-1}$.

Proof. (i) and (ii) see Lemma 1.1 and 1.2 in [22]. (iii) See Proposition 2.3 in [3]. ■

In the following we state some useful results on maximal monotone operators.

Lemma 2.2 *Let $T : \text{dom}(T) \subseteq \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ be a maximal monotone operator. Then,*

- (i) $\text{Gr}(T)$ is closed.
- (ii) T is bounded on bounded subsets of the interior of its domain.

Proof.

- (i) See Proposition 4.2.1(ii) in [8].

(ii) See Lemma 5(iii) in [5]. ■

Proposition 2.3 *Let $T : \text{dom}(T) \subseteq \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ be a point-to-set and maximal monotone operator. Given $\beta > 0$ then the operator $(I + \beta T)^{-1} : \mathbb{R}^n \rightarrow \text{dom}(T)$ is single valued and maximal monotone.*

Proof. See Theorem 4 in [18]. ■

Proposition 2.4 *Given $\beta > 0$ and $A : \text{dom}(A) \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a monotone operator and $B : \text{dom}(B) \subseteq \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ be a maximal monotone operator, then*

$$x = (I + \beta B)^{-1}(I - \beta A)(x),$$

if and only if, $0 \in (A + B)(x)$.

Proof. See Proposition 3.13 in [13]. ■

Now we define the so called Fejér convergence.

Definition 2.5 *Let S be a nonempty subset of \mathbb{R}^n . A sequence $\{x^k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n$ is said to be Fejér convergent to S , if and only if, for all $x \in S$ there exists $k_0 \geq 0$, such that $\|x^{k+1} - x\| \leq \|x^k - x\|$ for all $k \geq k_0$.*

This definition was introduced in [7] and have been further elaborated in [15] and [1]. A useful result on Fejér sequences is the following.

Proposition 2.6 *If $\{x^k\}_{k \in \mathbb{N}}$ is Fejér convergent to S , then:*

- (i) *the sequence $\{x^k\}_{k \in \mathbb{N}}$ is bounded;*
- (ii) *the sequence $\{\|x^k - x\|\}_{k \in \mathbb{N}}$ is convergent for all $x \in S$;*
- (iii) *if a cluster point x^* belongs to S , then the sequence $\{x^k\}_{k \in \mathbb{N}}$ converges to x^* .*

Proof. (i) and (ii) See Proposition 5.4 in [2]. (iii) See Theorem 5.5 in [2]. ■

3 The Algorithm

Let $A_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $B_i : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ be maximal monotone operators, with A_i point-to-point and B_i point-to-set, for all $i \in \mathbb{I}$. Assume that $\text{dom}(B_i) \subseteq \text{dom}(A_i)$, for all $i \in \mathbb{I} := \{1, 2, 3, \dots, m\}$ with $m \in \mathbb{N}$. Choose any nonempty, closed and convex set, $X \subseteq \cap_{i \in \mathbb{I}} \text{dom}(B_i)$, satisfying $X \cap S^* \neq \emptyset$. Thus, from now on, the solution set, S^* , is nonempty. Also we assume that the operators B_i for all $i \in \mathbb{I}$ satisfies, that for each bounded subset V of $\text{dom}(B_i)$ there exists $R > 0$, such that $B_i(x) \cap B[0, R] \neq \emptyset$, for all $x \in V$ and $i \in \mathbb{I}$ where $B[0, R]$ is the closed ball centered in 0 and radius R . We emphasize that this assumption holds trivially if $\text{dom}(B_i) = \mathbb{R}^n$ or $V \subset \text{int}(\text{dom}(B_i))$ or B_i is the normal cone in any subset of $\text{dom}(B_i)$.

Let $\{\beta_k\}_{k=0}^{\infty}$ be a sequence such that $\{\beta_k\}_{k \in \mathbb{N}} \subseteq [\check{\beta}, \hat{\beta}]$ with $0 < \check{\beta} \leq \hat{\beta} < \infty$, $\theta, \delta \in (0, 1)$, and be $\mathbb{I} = \{1, 2, 3, \dots, m\}$, $R > 0$ like assumption above. The algorithm is defined as follows:

Conceptual Algorithm A Let $\{\beta_k\}_{k \in \mathbb{N}}, \theta, \delta, R$ and \mathbb{I} like above.

Step 0 (Initialization): Take $x^0 \in X$.

Step 1 (Iterative Step 1): Given x^k , compute for all $i \in \mathbb{I}$,

$$J_i(x^k, \beta_k) := (I + \beta_k B_i)^{-1}(I - \beta_k A_i)(x^k). \quad (2)$$

Step 2 (Stopping Test 1): Define $\mathbb{I}_k^* := \{i \in \mathbb{I} : x^k = J_i(x^k, \beta_k)\}$. If $\mathbb{I}_k^* = \mathbb{I}$ stop.

Step 3 (Inner Loop): Otherwise, for all $i \in \mathbb{I} \setminus \mathbb{I}_k^*$ begin the inner loop over j . Put $j = 0$ and choose any $u_{(j,i)}^k \in B_i(\theta^j J_i(x^k, \beta_k) + (1 - \theta^j)x^k) \cap B[0, R]$. If

$$\left\langle A_i(\theta^j J_i(x^k, \beta_k) + (1 - \theta^j)x^k) + u_{(j,i)}^k, x^k - J_i(x^k, \beta_k) \right\rangle \geq \frac{\delta}{\beta_k} \|x^k - J_i(x^k, \beta_k)\|^2, \quad (3)$$

then $j_i(k) := j$ and stop. Else, $j = j + 1$.

Step 4 (Iterative Step 2): Set for all $i \in \mathbb{I} \setminus \mathbb{I}_k^*$

$$\alpha_{k,i} := \theta^{j_i(k)}, \quad (4)$$

$$\bar{u}_i^k := u_{j_i(k)}^k \quad (5)$$

$$\bar{x}_i^k := \alpha_{k,i} J_i(x^k, \beta_k) + (1 - \alpha_{k,i})x^k \quad (6)$$

and

$$x^{k+1} := \mathcal{F}_A(x^k). \quad (7)$$

Step 5 (Stop Criteria 2): If $x^{k+1} = x^k$ then stop. Otherwise, set $k \leftarrow k + 1$ and go to **Step 1**.

We consider three variants of this algorithm. Their main difference lies in the computation (7):

$$\mathcal{F}_{A.1}(x^k) = P_X(P_{H_k}(x^k)); \quad (\text{Variant A.1}) \quad (8)$$

$$\mathcal{F}_{A.2}(x^k) = P_{X \cap H_k}(x^k); \quad (\text{Variant A.2}) \quad (9)$$

$$\mathcal{F}_{A.3}(x^k) = P_{X \cap H_k \cap W(x^k)}(x^0); \quad (\text{Variant A.3}) \quad (10)$$

where

$$H_k := \bigcap_{i \in \mathbb{I} \setminus \mathbb{I}_k^*} H_i(\bar{x}_i^k, \bar{u}_i^k) \quad (11)$$

$$H_i(x, u) := \{y \in \mathbb{R}^n : \langle A_i(x) + u, y - x \rangle \leq 0\} \quad (12)$$

and

$$W(x) := \{y \in \mathbb{R}^n : \langle y - x, x^0 - x \rangle \leq 0\}. \quad (13)$$

This kind of hyperplane have been used in some works, see [4, 20].

4 Convergence Analysis

In this section we analyze the convergence of the algorithms presented in the previous section. First, we present some general properties as well as prove the well-definition of the conceptual algorithm.

Lemma 4.1 For all $(x, u) \in \text{Gr}(B_i)$, $S_i^* \subseteq H_i(x, u)$, for all $i \in \mathbb{I}$. Therefore $S^* \subset H_i(x, u)$ for all $i \in \mathbb{I}$.

Proof. Take $x^* \in S_i^*$. Using the definition of the solution, there exists $v^* \in B_i(x^*)$, such that $0 = A_i(x^*) + v^*$. By the monotonicity of $A_i + B_i$, we have

$$\langle A_i(x) + u - (A_i(x^*) + v^*), x - x^* \rangle \geq 0,$$

for all $(x, u) \in \text{Gr}(B_i)$. Hence,

$$\langle A_i(x) + u, x^* - x \rangle \leq 0$$

and by (12), $x^* \in H_i(x, u)$. ■

From now on, $\{x^k\}_{k \in \mathbb{N}}$ is the sequence generated by the conceptual algorithm.

Proposition 4.2 The conceptual algorithm is well-defined.

Proof. By Proposition 2.4, Stop Criteria 1 is well-defined. The proof of the well-definition of $j_i(k)$ is by contradiction. Fix $i \in \mathbb{I} \setminus \mathbb{I}_k^*$ and assume that for all $j \geq 0$ having chosen $u_{(j,i)}^k \in B_i(\theta^j J_i(x^k, \beta_k) + (1 - \theta^j)x^k) \cap B[0, R]$,

$$\left\langle A_i(\theta^j J_i(x^k, \beta_k) + (1 - \theta^j)x^k) + u_j^k, x^k - J_i(x^k, \beta_k) \right\rangle < \frac{\delta}{\beta_k} \|x^k - J_i(x^k, \beta_k)\|^2.$$

Since the sequence $\{u_{(j,i)}^k\}_{j=0}^\infty$ is bounded, there exists a subsequence $\{u_{(\ell_j,i)}^k\}_{j=0}^\infty$ of $\{u_{(j,i)}^k\}_{j=0}^\infty$, which converges to an element u_i^k belonging to $B_i(x^k)$ by maximality. Taking the limit over the subsequence $\{\ell_j\}_{j \in \mathbb{N}}$, we get

$$\langle \beta_k A_i(x^k) + \beta_k u_i^k, x^k - J_i(x^k, \beta_k) \rangle \leq \delta \|x^k - J_i(x^k, \beta_k)\|^2. \quad (14)$$

It follows from (2) that

$$\beta_k A_i(x^k) = x^k - J_i(x^k, \beta_k) - \beta_k v_i^k,$$

for some $v_i^k \in B_i(J_i(x^k, \beta_k))$.

Now, the above equality together with (14), lead to

$$\|x^k - J_i(x^k, \beta_k)\|^2 \leq \left\langle x^k - J_i(x^k, \beta_k) - \beta_k v_i^k + \beta_k u_i^k, x^k - J_i(x^k, \beta_k) \right\rangle \leq \delta \|x^k - J_i(x^k, \beta_k)\|^2,$$

using the monotonicity of B_i for the first inequality. So,

$$(1 - \delta) \|x^k - J_i(x^k, \beta_k)\|^2 \leq 0,$$

which contradicts that $i \in \mathbb{I} \setminus \mathbb{I}_k^*$. Thus, the conceptual algorithm is well-defined. ■

Proposition 4.3 $x^k \in H_k$ if and only if, $x^k \in S^*$.

Proof. If $x^k \in H_k$ then $x^k \in H_i(\bar{x}_i^k, \bar{u}_i^k)$ for all $i \in \mathbb{I} \setminus \mathbb{I}_k^*$ by definition of H_k . Now by Proposition (4.2) of [6] we have that $x^k \in S_i^*$ for all $i \in \mathbb{I}$, then $x^k \in S^*$. Conversely, if $x^k \in S^*$ then $x^k \in S_i^*$ then $x^k \in H_i(\bar{x}_i^k, \bar{u}_i^k)$ for all $i \in \mathbb{I}$ using the same proposition, implying that $x^k \in H_k$. ■

Finally, a useful algebraic property on the sequence generated by the conceptual algorithm, which is a direct consequence of the inner loop and (6).

Corollary 4.4 Let $\{x^k\}_{k \in \mathbb{N}}$, $\{\beta_k\}_{k \in \mathbb{N}}$ and $\{\alpha_{(k,i)}\}_{k \in \mathbb{N}}$ be sequences generated by the conceptual algorithm. With δ and $\hat{\beta}$ as in the conceptual algorithm. Then,

$$\langle A_i(\bar{x}_i^k) + \bar{u}_i^k, x^k - \bar{x}_i^k \rangle \geq \frac{\alpha_{k,i}\delta}{\hat{\beta}} \|x^k - J_i(x^k, \beta_k)\|^2 \geq 0, \quad (15)$$

for all k .

4.1 Convergence analysis of Variant A.1

In this section, all results are for **Variant A.1**, which is summarized below.

Variant A.1 $x^{k+1} = \mathcal{F}_{A.1}(x^k) = P_X(P_{H_k}(x^k))$

Proposition 4.5 If **Variant A.1** stops, then $x^k \in S^*$.

Proof. If Stop Criteria 2 is satisfied, $x^{k+1} = P_X(P_{H_k}(x^k)) = x^k$. Using Proposition 2.1(ii), we have

$$\langle P_{H_k}(x^k) - x^k, z - x^k \rangle \leq 0, \quad (16)$$

for all $z \in X$. Now using Proposition 2.1(ii),

$$\langle P_{H_k}(x^k) - x^k, P_{H_k}(x^k) - z \rangle \leq 0, \quad (17)$$

for all $z \in H_k$. Since $X \cap H_k \neq \emptyset$ summing (16) and (17), with $z \in X \cap H_k$, we get

$$\|x^k - P_{H_k}(x^k)\|^2 = 0.$$

Hence, $x^k = P_{H_k}(x^k)$, implying that $x^k \in H_k$ and by Proposition 4.3, $x^k \in S^*$. ■

Proposition 4.6 (i) The sequence $\{x^k\}_{k \in \mathbb{N}}$ is Fejér convergente to $S^* \cap X$.

(ii) The sequence $\{x^k\}_{k \in \mathbb{N}}$ is bounded.

(iii) $\lim_{k \rightarrow \infty} \|P_{H_k}(x^k) - x^k\|^2 = 0$.

(iv) $\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\|^2 = 0$.

Proof. (i) Take $x^* \in S^* \cap X$. Using (8), Proposition 2.1(i) and Lemma 4.1, we have

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &= \|P_X(P_{H_k}(x^k)) - P_X(P_{H_k}(x^*))\|^2 \leq \|P_{H_k}(x^k) - P_{H_k}(x^*)\|^2 \\ &\leq \|x^k - x^*\|^2 - \|P_{H_k}(x^k) - x^k\|^2. \end{aligned} \quad (18)$$

So, $\|x^{k+1} - x^*\| \leq \|x^k - x^*\|$. (ii) Follows immediately from item (i). (iii) Take $x^* \in S^* \cap X$. Using (18) yields

$$\|P_{H_k}(x^k) - x^k\|^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2. \quad (19)$$

Now using Proposition 2.6 and item (i) we have that the right side of equation (19) go to zero. Obtaining the result. (iv) Since the sequence $\{x^k\}_{k \in \mathbb{N}}$ belong to X , we have,

$$\|x^{k+1} - x^k\|^2 = \|P_X(P_{H_k}(x^k)) - P_X(x^k)\|^2 \leq \|P_{H_k}(x^k) - x^k\|^2.$$

Taking limits in the above equation and using the previous item we have the result. ■

Proposition 4.7 For all $i \in \mathbb{I}$ we have,

$$\lim_{k \rightarrow \infty} \langle A_i(\bar{x}_i^k) + \bar{u}_i^k, x^k - \bar{x}_i^k \rangle = 0.$$

Proof. For all $i \in \mathbb{I}$. Using Proposition 2.1(i) and the fact that $H_k \subset H(\bar{x}_i^k, \bar{u}_i^k)$ by (11), we have that,

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &= \|P_X(P_{H_k}(x^k)) - P_X(x^*)\|^2 \leq \|P_{H_k}(x^k) - x^*\|^2 \\ &= \|P_{H_k}(x^k) - P_{H(\bar{x}_i^k, \bar{u}_i^k)}(x^k) + P_{H(\bar{x}_i^k, \bar{u}_i^k)}(x^k) - x^*\|^2 \\ &\leq \|P_{H_k}(x^k) - x^k\|^2 + \|P_{H(\bar{x}_i^k, \bar{u}_i^k)}(x^k) - x^*\|^2. \end{aligned} \quad (20)$$

Now using Proposition 2.1(i) and reordering (20), we get,

$$\|P_{H(\bar{x}_i^k, \bar{u}_i^k)}(x^k) - x^k\|^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 + \|P_{H_k}(x^k) - x^k\|^2.$$

Using the fact that,

$$P_{H(\bar{x}_i^k, \bar{u}_i^k)}(x^k) = x^k - \frac{\langle A_i(\bar{x}_i^k) + \bar{u}_i^k, x^k - \bar{x}_i^k \rangle}{\|A_i(\bar{x}_i^k) + \bar{u}_i^k\|^2} (A_i(\bar{x}_i^k) + \bar{u}_i^k),$$

and the previous equation, we have,

$$\frac{(\langle A_i(\bar{x}_i^k) + \bar{u}_i^k, x^k - \bar{x}_i^k \rangle)^2}{\|A_i(\bar{x}_i^k) + \bar{u}_i^k\|^2} \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 + \|P_{H_k}(x^k) - x^k\|^2. \quad (21)$$

By Proposition 2.3 and the continuity of A_i we have that J_i is continuo, since $\{x^k\}_{k \in \mathbb{N}}$ and $\{\beta_k\}_{k \in \mathbb{N}}$ are bounded then $\{J_i(x^k, \beta_k)\}_{k \in \mathbb{N}}$ and $\{\bar{x}_i^k\}_{k \in \mathbb{N}}$ are bounded, implying the boundedness of $\{\|A_i(\bar{x}_i^k) + \bar{u}_i^k\|\}_{k \in \mathbb{N}}$ for all $i \in \mathbb{I}$.

Using Proposition 2.6(ii) and (iii), the right side of (21) goes to 0, when k goes to ∞ , establishing the result. \blacksquare

Next we establish our main convergence result on **Variant A.1**.

Theorem 4.8 The sequence $\{x^k\}_{k \in \mathbb{N}}$ converges to some element belonging to $S^* \cap X$.

Proof. We claim that there exists a cluster point of $\{x^k\}_{k \in \mathbb{N}}$ belonging to S^* . The existence of the cluster points follows from Proposition 4.6(ii). Let $\{x^{j_k}\}_{k \in \mathbb{N}}$ be a convergent subsequence of $\{x^k\}_{k \in \mathbb{N}}$ such that, for all $i \in \mathbb{I}$ the sequences $\{\bar{x}_i^{j_k}\}_{k \in \mathbb{N}}$, $\{\bar{u}_i^{j_k}\}_{k \in \mathbb{N}}$, $\{\alpha_{j_k, i}\}_{k \in \mathbb{N}}$ and $\{\beta_{j_k}\}_{k \in \mathbb{N}}$ are convergents, and $\lim_{k \rightarrow \infty} x^{j_k} = \tilde{x}$.

Using Proposition 4.6(iii) and taking limits in (15) over the subsequence $\{j_k\}_{k \in \mathbb{N}}$, we have for all $i \in \mathbb{I}$,

$$0 = \lim_{k \rightarrow \infty} \langle A_i(\bar{x}_i^{j_k}) + \bar{u}_i^{j_k}, x^{j_k} - \bar{x}_i^{j_k} \rangle \geq \lim_{k \rightarrow \infty} \frac{\alpha_{j_k, i} \delta}{\hat{\beta}} \|x^{j_k} - J_i(x^{j_k}, \beta_{j_k})\|^2 \geq 0. \quad (22)$$

Therefore,

$$\lim_{k \rightarrow \infty} \alpha_{j_k, i} \|x^{j_k} - J_i(x^{j_k}, \beta_{j_k})\| = 0.$$

Now consider the two possible cases.

(a) First, assume that $\lim_{k \rightarrow \infty} \alpha_{j_k, i} \neq 0$, i.e., $\alpha_{j_k, i} \geq \bar{\alpha}$ for all k and some $\bar{\alpha} > 0$. In view of (22),

$$\lim_{k \rightarrow \infty} \|x^{j_k} - J_i(x^{j_k}, \beta_{j_k})\| = 0. \quad (23)$$

Since J_i is continuous, by the continuity of A_i and $(I + \beta_k B_i)^{-1}$ and by Proposition 2.3, (23) becomes

$$\tilde{x} = J_i(\tilde{x}, \tilde{\beta}),$$

which implies that $\tilde{x} \in S_i^*$ for all $i \in \mathbb{I}$. Then $\tilde{x} \in S^*$ establishing the claim.

(b) On the other hand, if $\lim_{k \rightarrow \infty} \alpha_{j_k, i} = 0$ then for $\theta \in (0, 1)$ as in the conceptual algorithm, we have

$$\lim_{k \rightarrow \infty} \frac{\alpha_{j_k, i}}{\theta} = 0.$$

Define

$$y_i^{j_k} := \frac{\alpha_{j_k, i}}{\theta} J_i(x^{j_k}, \beta_{j_k}) + \left(1 - \frac{\alpha_{j_k, i}}{\theta}\right) x^{j_k}.$$

Then,

$$\lim_{k \rightarrow \infty} y_i^{j_k} = \tilde{x}. \quad (24)$$

Using the definition of the $j_i(k)$ and (4), we have that $y_i^{j_k}$ does not satisfy (3) implying

$$\left\langle A_i(y_i^{j_k}) + u_{j_i(k)-1}^{j_k} - \frac{\delta}{\beta_k} (x^k - J_i(x^k, \beta_k)), x^k - J_i(x^k, \beta_k) \right\rangle > 0,$$

equivalent to

$$\left\langle A_i(y_i^{j_k}) + u_{j_i(k)-1, i}^{j_k}, x^k - J_i(x^k, \beta_k) \right\rangle > \frac{\delta}{\beta_k} \|x^k - J_i(x^k, \beta_k)\|^2, \quad (25)$$

for $u_{j_i(k)-1, i}^{j_k} \in B_i(y_i^{j_k})$ and all $k \in \mathbb{N}$ and $i \in \mathbb{I}$.

Redefining the subsequence $\{j_k\}_{k \in \mathbb{N}}$, if necessary, we may assume that $\{u_{j_i(k)-1, i}^{j_k}\}_{k \in \mathbb{N}}$ converges to \tilde{u}_i . By the maximality of B_i , \tilde{u}_i belongs to $B_i(\tilde{x})$. Using the continuity of J_i , $\{J(x^{j_k}, \beta_{j_k})\}_{k \in \mathbb{N}}$ converges to $J_i(\tilde{x}, \tilde{\beta})$. Using (24) and taking limit in (25) over the subsequence $\{j_k\}_{k \in \mathbb{N}}$, we have

$$\left\langle A_i(\tilde{x}) + \tilde{u}_i, \tilde{x} - J_i(\tilde{x}, \tilde{\beta}) \right\rangle \leq \frac{\delta}{\tilde{\beta}} \|\tilde{x} - J_i(\tilde{x}, \tilde{\beta})\|^2. \quad (26)$$

Using (2) and multiplying by $\tilde{\beta}$ on both sides of (26), we get

$$\langle \tilde{x} - J_i(\tilde{x}, \tilde{\beta}) - \tilde{\beta} \tilde{v}_i + \tilde{\beta} \tilde{u}_i, \tilde{x} - J_i(\tilde{x}, \tilde{\beta}) \rangle \leq \delta \|\tilde{x} - J_i(\tilde{x}, \tilde{\beta})\|^2,$$

where $\tilde{v}_i \in B_i(J_i(\tilde{x}, \tilde{\beta}))$. Applying the monotonicity of B_i , we obtain

$$\|\tilde{x} - J_i(\tilde{x}, \tilde{\beta})\|^2 \leq \delta \|\tilde{x} - J_i(\tilde{x}, \tilde{\beta})\|^2,$$

implying that $\|\tilde{x} - J_i(\tilde{x}, \tilde{\beta})\| \leq 0$. Thus, $\tilde{x} = J_i(\tilde{x}, \tilde{\beta})$ and hence, $\tilde{x} \in S_i^*$ for all $i \in \mathbb{I}$, thus $\tilde{x} \in S^*$. ■

4.2 Convergence analysis of Variant A.2

In this section, all results are for **Variant A.2**, which is summarized below.

Variant A.2 $x^{k+1} = \mathcal{F}_{A.2}(x^k) = P_{X \cap H_k}(x^k)$

Proposition 4.9 *If Variant A.2 stops, then $x^k \in S^*$.*

Proof. If $x^{k+1} = P_{X \cap H_k}(x^k) = x^k$ then $x^k \in X \cap H_k$ and by Proposition 4.3, $x^k \in S^* \cap X$. ■
From now on assume that **Variant A.2** does not stop.

Proposition 4.10 *The sequence $\{x^k\}_{k \in \mathbb{N}}$ is Féjer convergent to $S^* \cap X$. Moreover, it is bounded and*

$$\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0.$$

Proof. Take $x^* \in S^* \cap X$. By Lemma 4.1, $x^* \in H_k \cap X$, for all k . Then using Proposition 2.1(ii) and (9)

$$\|x^{k+1} - x^*\|^2 - \|x^k - x^*\|^2 + \|x^{k+1} - x^k\|^2 = 2\langle x^* - x^{k+1}, x^k - x^{k+1} \rangle \leq 0,$$

we obtain

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^k\|^2. \quad (27)$$

The above inequality implies that $\{x^k\}_{k \in \mathbb{N}}$ is Féjer convergent to $S^* \cap X$. Hence by Proposition 2.6(i) and (ii), $\{x^k\}_{k \in \mathbb{N}}$ is bounded and thus $\{\|x^k - x^*\|\}_{k \in \mathbb{N}}$ is a convergent sequence. Taking limits in (27), we get

$$\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0. \quad \blacksquare$$

The next proposition shows a relation between the projection steps in **Variant A.1** and **A.2**. This fact has a geometry interpretation, since the projection of **Variant A.2** is done over a small set, improving the convergence of **Variant A.1**. Note that this can be reduce the number of iterations, avoiding possible zigzagging of **Variant A.1**.

Proposition 4.11 *Let $\{x^k\}_{k \in \mathbb{N}}$ the sequence generated by Variant A.2. Then,*

- (i) $x^{k+1} = P_{X \cap H_k}(P_{H_k}(x^k))$.
- (ii) *For all $i \in \mathbb{I}$ we have, $\lim_{k \rightarrow \infty} \langle A_i(\bar{x}_i^k) + \bar{u}_i^k, x^k - \bar{x}_i^k \rangle = 0$.*

Proof. (i) Fix any $y \in X \cap H_k$. Since $x^k \in X$ but $x^k \notin H_k$ by Proposition 4.3, there exists $\gamma \in [0, 1]$, such that $\tilde{x} = \gamma x^k + (1 - \gamma)y \in X \cap \partial H_k$. Hence,

$$\begin{aligned} \|y - P_{H_k}(x^k)\|^2 &\geq (1 - \gamma)^2 \|y - P_{H_k}(x^k)\|^2 \\ &= \|\tilde{x} - \gamma x^k - (1 - \gamma)P_{H_k}(x^k)\|^2 \\ &= \|\tilde{x} - P_{H_k}(x^k)\|^2 + \gamma^2 \|x^k - P_{H_k}(x^k)\|^2 - 2\gamma \langle \tilde{x} - P_{H_k}(x^k), x^k - P_{H_k}(x^k) \rangle \\ &\geq \|\tilde{x} - P_{H_k}(x^k)\|^2, \end{aligned} \quad (28)$$

where the last inequality follows from Proposition 2.1(ii), applied with $X = H_k$, $x = x^k$ and $z = \tilde{x} \in H_k$. Furthermore, we have

$$\begin{aligned}
\|\tilde{x} - P_{H_k}(x^k)\| &\geq \|\tilde{x} - x^k\| - \|x^k - P_{H_k}(x^k)\| \\
&\geq \|x^{k+1} - x^k\| - \|x^k - P_{H_k}(x^k)\| \\
&\geq \|x^{k+1} - x^k\| \\
&\geq \|x^{k+1} - P_{H_k}(x^k)\|,
\end{aligned} \tag{29}$$

where the first equality follows by the triangle inequality, using the fact that $\tilde{x} \in X \cap H_k$ and $x^{k+1} = P_{X \cap H_k}(x^k)$ in the second inequality, the third one is trivial, and the last one inequality by the fact that $x^{k+1} \in H_k$ and Proposition 2.1(i) with $X = H_k$. Combining (28) and (29), we obtain

$$\|y - P_{H_k}(x^k)\| \geq \|x^{k+1} - P_{H_k}(x^k)\|,$$

for all $y \in X \cap H_k$. Hence, $x^{k+1} = P_{X \cap H_k}(P_{H_k}(x^k))$.

(ii) Take $x^* \in X \cap S^*$. By item (i), Lemma 4.1 and Proposition 2.1(i), we have

$$\|x^{k+1} - x^*\|^2 = \|P_{X \cap H_k}(P_{H_k}(x^k)) - P_{X \cap H_k}(x^*)\|^2 \leq \|P_{H_k}(x^k) - x^*\|^2.$$

The proof is similar to the proof of Proposition 4.7. ■

Finally we present the convergence result for **Variant A.2**.

Theorem 4.12 *The sequence $\{x^k\}_{k \in \mathbb{N}}$ converges to some point belonging to $S^* \cap X$.*

Proof. Repeat the proof of Theorem 4.8. ■

4.3 Convergence analysis of Variant A.3

In this section, all results are for **Variant A.3**, which is summarized below.

Variant A.3 $x^{k+1} = \mathcal{F}_{A.3}(x^k) = P_{X \cap H_k \cap W(x^k)}(x^0)$

Proposition 4.13 *If Variant A.3 stops, then $x^k \in S^* \cap X$.*

Proof. If Stop Criteria 2 is satisfied then, $x^{k+1} = P_{X \cap H_k \cap W_k}(x^0) = x^k$. So, $x^k \in X \cap H_k \cap W_k \subset X \cap H_k$ and finally using Proposition 4.3, $x^k \in S^* \cap X$. ■

From now on we assume that **Variant A.3** does not stop. Observe that, in virtue of their definitions, W_k and H_k are convex and closed sets, for each k . Therefore $X \cap H_k \cap W_k$ is a convex and closed set. So, if $X \cap H_k \cap W_k$ is nonempty, then the next iterate, x^{k+1} , is well-defined. The following lemma guarantees this fact.

Lemma 4.14 $S^* \cap X \subset H_k \cap W_k$, for all k .

Proof. We proceed by induction. By definition, $S^* \cap X \neq \emptyset$. By Lemma 4.1, $S^* \cap X \subset H_k$, for all k . For $k = 0$, as $W_0 = \mathbb{R}^n$, $S^* \cap X \subset H_0 \cap W_0$.

Assume that $S^* \cap X \subset H_\ell \cap W_\ell$, for $\ell \leq k$. Henceforth, $x^{k+1} = P_{X \cap H_k \cap W_k}(x^0)$ is well-defined. Then, by Proposition 2.1(ii), we have

$$\langle x^* - x^{k+1}, x^0 - x^{k+1} \rangle = \langle x^* - P_{X \cap H_k \cap W_k}(x^0), x^0 - P_{X \cap H_k \cap W_k}(x^0) \rangle \leq 0, \quad (30)$$

for all $x^* \in S^* \cap X$. The inequality follows by the induction hypothesis. Now, (30) implies that $x^* \in W_{k+1}$ and hence, $S^* \cap X \subset H_{k+1} \cap W_{k+1}$. ■

The above lemma shows that the set $X \cap H_k \cap W_k$ is nonempty and in consequence the projection step, given in (10), is well-defined.

Corollary 4.15 Variant A.3 *is well-defined.*

Proof. By Lemma 4.14, $S^* \cap X \subset H_k \cap W_k$, for all k . Then, given x^0 , the sequence $\{x^k\}_{k \in \mathbb{N}}$ is computable. ■

Before proving the convergence of the sequence, we study its boundedness. The next lemma shows that the sequence remains in a ball determined by the initial point.

Lemma 4.16 *The sequence $\{x^k\}_{k \in \mathbb{N}}$ is bounded. Furthermore,*

$$\{x^k\}_{k \in \mathbb{N}} \subset B \left[\frac{1}{2}(x^0 + \bar{x}), \frac{1}{2}\rho \right] \cap X,$$

where $\bar{x} = P_{S^* \cap X}(x^0)$ and $\rho = \text{dist}(x^0, S^* \cap X)$.

Proof. $S^* \cap X \subset H_k \cap W_k$ follows from Lemma 4.14. Moreover, from (10), we obtain that

$$\|x^{k+1} - x^0\| \leq \|z - x^0\|, \quad (31)$$

for all k and all $z \in S^* \cap X$. Henceforth, taking $z = \bar{x}$ in (31),

$$\|x^{k+1} - x^0\| \leq \|\bar{x} - x^0\| = \rho, \quad (32)$$

for all k . Thus, $\{x^k\}_{k \in \mathbb{N}}$ is bounded. Define $z^k = x^k - \frac{1}{2}(x^0 + \bar{x})$ and $\bar{z} = \bar{x} - \frac{1}{2}(x^0 + \bar{x})$. It follows from the fact $\bar{x} \in W_{k+1}$, that

$$\begin{aligned} 0 &\geq 2\langle \bar{x} - x^{k+1}, x^0 - x^{k+1} \rangle \\ &= 2\left\langle \bar{z} + \frac{1}{2}(x^0 + \bar{x}) - z^{k+1} - \frac{1}{2}(x^0 + \bar{x}), z^0 + \frac{1}{2}(x^0 + \bar{x}) - z^{k+1} - \frac{1}{2}(x^0 + \bar{x}) \right\rangle \\ &= 2\left\langle \bar{z} - z^{k+1}, z^0 - z^{k+1} \right\rangle = \left\langle \bar{z} - z^{k+1}, -\bar{z} - z^{k+1} \right\rangle = \|z^{k+1}\|^2 - \|\bar{z}\|^2, \end{aligned}$$

where we have used that $\bar{z} = -z^0$ in the third equality. So,

$$\left\| x^{k+1} - \frac{x^0 + \bar{x}}{2} \right\| \leq \left\| \bar{x} - \frac{x^0 + \bar{x}}{2} \right\| = \frac{\rho}{2},$$

for all k . Now, the result follows from the feasibility of $\{x^k\}_{k \in \mathbb{N}}$, which, in turn, is a consequence of (10). ■

Now, we focus on the properties of the accumulation points.

Lemma 4.17 *All accumulation points of $\{x^k\}_{k \in \mathbb{N}}$ belong to $S^* \cap X$.*

Proof. Since $x^{k+1} \in W_k$,

$$0 \geq 2\langle x^{k+1} - x^k, x^0 - x^k \rangle = \|x^{k+1} - x^k\|^2 - \|x^{k+1} - x^0\|^2 + \|x^k - x^0\|^2.$$

Equivalently

$$0 \leq \|x^{k+1} - x^k\|^2 \leq \|x^{k+1} - x^0\|^2 - \|x^k - x^0\|^2,$$

establishing that the sequence $\{\|x^k - x^0\|\}_{k \in \mathbb{N}}$ is monotone and nondecreasing. From Lemma 4.16, we get that $\{\|x^k - x^0\|\}_{k \in \mathbb{N}}$ is bounded, and thus, convergent. Therefore,

$$\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0. \quad (33)$$

Since $x^{k+1} \in H_k$, we get for all $i \in \mathbb{I}$ that,

$$\langle A_i(\bar{x}_i^k) + \bar{u}_i^k, x^{k+1} - \bar{x}_i^k \rangle \leq 0, \quad (34)$$

with \bar{u}_i^k and \bar{x}_i^k as (5) and (6).

Using (6) and (34), we have

$$\langle A_i(\bar{x}_i^k) + \bar{u}_i^k, x^{k+1} - x^k \rangle + \alpha_{k,i} \langle A_i(\bar{x}_i^k) + \bar{u}_i^k, x^k - J_i(x^k, \beta_k) \rangle \leq 0.$$

Combining the above inequality with the stop criteria of Inner Loop, given in (3), we get for all $i \in \mathbb{I}$

$$\langle A_i(\bar{x}_i^k) + \bar{u}_i^k, x^{k+1} - x^k \rangle + \frac{\alpha_{k,i} \delta}{\hat{\beta}} \|x^k - J_i(x^k, \beta_k)\|^2 \leq 0. \quad (35)$$

Choosing a subsequence $\{j_k\}_{k \in \mathbb{N}}$ such that the subsequences $\{x^{j_k}\}_{k \in \mathbb{N}}$, $\{\beta_{j_k}\}_{k \in \mathbb{N}}$ and $\{\bar{u}_i^{j_k}\}_{k \in \mathbb{N}}$ converge to \tilde{x} , $\tilde{\beta}$ and \tilde{u}_i respectively. This is possible by the boundedness of $\{\bar{u}_i^k\}_{k \in \mathbb{N}}$, by hypothesis on B_i , bounded of $\{x^k\}_{k \in \mathbb{N}}$ and $\{\beta_k\}_{k \in \mathbb{N}}$. Taking limits in (35), we have

$$\lim_{k \rightarrow \infty} \alpha_{j_k,i} \|x^{j_k} - J_i(x^{j_k}, \beta_{j_k})\|^2 = 0. \quad (36)$$

Now we consider two cases, $\lim_{k \rightarrow \infty} \alpha_{j_k,i} = 0$ or $\lim_{k \rightarrow \infty} \alpha_{j_k,i} \neq 0$ (taking a subsequence again if necessary).

(a) $\lim_{k \rightarrow \infty} \alpha_{j_k,i} \neq 0$, i.e., for all $i \in \mathbb{I}$, $\alpha_{j_k,i} \geq \tilde{\alpha}_i$ for all k and some $\tilde{\alpha}_i > 0$. By (36),

$$\lim_{k \rightarrow \infty} \|x^{j_k} - J(x^{j_k}, \beta_{j_k})\|^2 = 0.$$

By continuity of J_i , we have $\tilde{x} = J_i(\tilde{x}, \tilde{\beta})$ and hence by Proposition 2.4, $\tilde{x} \in S_i^*$ for all $i \in \mathbb{I}$, therefor $\tilde{x} \in S_*$.

(b) $\lim_{k \rightarrow \infty} \alpha_{j_k,i} = 0$, then $\lim_{k \rightarrow \infty} \frac{\alpha_{j_k,i}}{\theta} = 0$. It follows in the same, manner as in the proof of Theorem 4.8(b). \blacksquare

Finally, we are ready to prove the convergence of the sequence $\{x^k\}_{k \in \mathbb{N}}$ generated by **Variant A.3**, to the solution closest to x^0 .

Theorem 4.18 *Define $\bar{x} = P_{S^* \cap X}(x^0)$. Then, $\{x^k\}_{k \in \mathbb{N}}$ converges to \bar{x} .*

Proof. By Lemma 4.16, $\{x^k\}_{k \in \mathbb{N}} \subset B\left[\frac{1}{2}(x^0 + \bar{x}), \frac{1}{2}\rho\right] \cap X$, so it is bounded. Let $\{x^{j_k}\}_{k \in \mathbb{N}}$ be a convergent subsequence of $\{x^k\}_{k \in \mathbb{N}}$, and let \hat{x} be its limit. Evidently $\hat{x} \in B\left[\frac{1}{2}(x^0 + \bar{x}), \frac{1}{2}\rho\right] \cap X$. Furthermore, by Lemma 4.17, $\hat{x} \in S^* \cap X$. Then,

$$\hat{x} \in S^* \cap X \cap B\left[\frac{1}{2}(x^0 + \bar{x}), \frac{1}{2}\rho\right] = \{\bar{x}\},$$

implying that $\hat{x} = \bar{x}$, hence \bar{x} is the unique limit point of $\{x^k\}_{k \in \mathbb{N}}$. Thus, $\{x^k\}_{k \in \mathbb{N}}$ converges to $\bar{x} \in S^* \cap X$. ■

5 Conclusions

In this paper, we present a variant of forward-backward splitting methods for solving a system of inclusion problems composed by the sum of two operators. A conceptual algorithm have been proposed containing three variants with different projections steps. A linesearch, for relax the hypothesis of Lipschitz continuity on forwards operators, have been proposed. The convergence analyse of three variant are discussed. The results presented here, improve the previous in the literature by relaxing the hypothesis.

References

- [1] Bauschke, H.H., Borwein, J.M. On projection algorithms for solving convex feasibility problems. *SIAM Review* **38** (1996) 367-426.
- [2] Bauschke, H.H., Combettes, Patrick L. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. Springer, (2011).
- [3] Bauschke, H. H., Burke, J. V., Deutsch, F. R., Hundal, H. S., Vanderwerff, J. D. A new proximal point iteration that converges weakly but not in norm. *Proceedings of the American Mathematical Society* **133** (2005) 1829-1835.
- [4] Bello Cruz, J.Y., Iusem, A.N. A strongly convergent method for nonsmooth convex minimization in Hilbert spaces. *Numerical Functional Analysis and Optimization* **32** (2011) 1009 -1018.
- [5] Bello Cruz, J.Y., Iusem, A.N. Convergence of direct methods for paramonotone variational inequalities. *Computation Optimization and Applications* **46** (2010) 247-263.
- [6] Bello Cruz, J.Y., Díaz Millán, R. A variant of forward-backward splitting method for the sum of two monotone operators with a new search strategy. *Optimization* DOI:10.1080/02331934.2014.883510 (2014).
- [7] Browder, F.E. Convergence theorems for sequences of nonlinear operators in Banach spaces. *Mathematische Zeitschrift* **100** (1967) 201-225.
- [8] Burachik, R.S., Iusem, A.N. *Set-Valued Mappings and Enlargements of Monotone Operators*. Springer, Berlin (2008).

- [9] Y. Censor, A. Gibali, and S. Reich. A von Neumann alternating method for finding common solutions to variational inequalities. *Nonlinear Analysis Series A: Theory, Methods and Applications* **75**, (2012) 4596-4603.
- [10] Y. Censor, A. Gibali, S. Reich, and S. Sabach. Common solutions to variational inequalities. *Set-Valued and Variational Analysis* **20**, (2012) 229247.
- [11] Y. Censor, A. Gibali, and S. Reich. Algorithms for the split variational inequality problem. *Numerical Algorithms* **59**, (2012) 301323.
- [12] Díaz Millán, R. *On several algorithms for variational inequality and inclusion problems*. PhD thesis, Federal University of Goiás, Goiânia, GO, 2015. Institute of Mathematic and Statistic, IME-UFG.
- [13] Eckstein, J. *Splitting Methods for Monotone Operators, with Applications to Parallel Optimization*. PhD thesis, Massachusetts Institute of Technology, Cambridge, MA, 1989. Report LIDS-TH-1877, Laboratory for Information and Decision Systems, M.I.T.
- [14] Eslamian, M., Saejung, S., Vahidi, J. Common solutions of a system of variational inequality problems. *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics* **77** Iss.1 (2015).
- [15] Iusem, A.N., Svaiter, B.F., Teboulle, M. Entropy-like proximal methods in convex programming. *Mathematics of Operations Research* **19** (1994) 790-814.
- [16] Konnov, I.V.: On systems of variational inequalities. *Russian Mathematics*, **41**, No. 12, (1997) 79-88.
- [17] Konnov, I.V.: Splitting-type method for systems of variational inequalities. *Computer and Operations Research* **33**, (2006) 520-534.
- [18] Minty, G. Monotone (nonlinear) operators in Hilbert Space. *Duke Mathematical Journal* **29** (1962) 341-346.
- [19] Semenov, V.V. Hybrid splitting methods for the system of operator inclusions with monotone operators. *Cybernetics and Systems Analysis* **50** (2014) 741-749.
- [20] Solodov, M.V., Svaiter, B.F. Forcing strong convergence of proximal point iterations in a Hilbert space. *Mathematical Programming* **87** (2000) 189-202.
- [21] Tseng, P. A modified forward-backward splitting method for maximal monotone mappings. *SIAM on Journal Control Optimization* **38** (2000) 431-446.
- [22] Zaraytonelo, E.H.: Projections on convex sets in Hilbert space and spectral theory. in Contributions to Nonlinear Functional Analysis, E. Zarantonello, ed., Academic Press, New York (1971) 237-424.